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# The shape of a tridiagonal pair(Algebraic combinatorics and the related areas of research)

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# The shape of a tridiagonal pair

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We announce that the shape conjecture is proved in general for the case where the parameter  $q$  is not a root of unity.

Let  $V$  be a finite dimensional vector space over  $\mathbf{C}$  and  $A, A^*$  semisimple linear transformations of  $V$ . Let  $V = \bigoplus_{i=0}^d V_i = \bigoplus_{i=0}^{d^*} V_i^*$  denote the eigenspace decompositions of  $V$  for  $A, A^*$ , respectively. The pair  $A, A^*$  is called a TD-pair (tridiagonal pair) if there exist an ordering of  $V_i$  and that of  $V_i^*$  such that

$$\begin{aligned} (1) \quad & AV_i^* \subset V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq d^*), \\ & A^*V_i \subset V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \end{aligned}$$

where  $V_{-1} = V_{d+1} = V_{-1}^* = V_{d^*+1}^* = 0$ , and

$$(2) \quad V \text{ is irreducible as an } \langle A, A^* \rangle\text{-module.}$$

A TD-pair arises from a P- and Q-polynomial association scheme with  $A, A^*$  the standard generators of the Terwilliger algebra that describe the P- and Q-polynomial structures of the association scheme, and with  $V$  an irreducible submodule of the standard module of the Terwilliger algebra.

In [1], it is shown that

$$\begin{aligned} (i) \quad & d = d^*, \\ (ii) \quad & \rho_i = \dim V_i = \dim V_{d-i} = \dim V_i^* = \dim V_{d^*-i}^* \quad (0 \leq i \leq d), \end{aligned}$$

and the sequence  $\rho_i$  is conjectured to satisfy

$$(iii) \quad (\text{the shape conjecture}) \quad \rho_i \leq \binom{d}{i}.$$

We announce that the shape conjecture is true in the case where the parameter  $q$  is not a root of unity. The proof will be published in a subsequent paper.

A TD-pair satisfies the TD-relations (tridiagonal relations) [1]:

$$\begin{aligned} \text{(TD)} \quad & [A, [A, [A, A^*]_{q^{-1}}]_q] = [\gamma A^2 + \delta A, A^*], \\ & [A^*, [A^*, [A^*, A]_{q^{-1}}]_q] = [\gamma^* A^{*2} + \delta^* A^*, A] \end{aligned}$$

for some  $\gamma, \gamma^*, \delta, \delta^* \in \mathbf{C}$ , where  $[x, y] = xy - yx$ ,  $[x, y]_q = qxy - q^{-1}yx$ . By applying an affine transformation to  $A$  and another one to  $A^*$ , we can normalize the TD-relations to have  $\gamma = \gamma^* = 0$  when  $q \neq 1$ . Such TD-relations are regarded as a  $q$ -analogue of the Dolan-Grady relations that define the Onsager algebra. The shape conjecture is proved by analysing the infinite dimensional algebra generated by two symbols over  $\mathbf{C}$  subject to the  $q$ -Dolan-Grady relations. In the case of  $\delta = \delta^* = 0$ , the  $q$ -Dolan-Grady relations are the  $q$ -Serre relations and the shape conjecture is already proved in [2].

## References

- [1] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to  $P$ - and  $Q$ -polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; [arXiv:math.CO/0406556](#).
- [2] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. *J. Pure Appl. Algebra* **188** (2004) 145–160; [arXiv:math.QA/0304244](#).